PARABOLIC GEOMETRIES FOR PEOPLE THAT LIKE PICTURES

LECTURE 10: THE CONFORMAL SPHERE

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Last time, we briefly explored projective geometry, which we thought of as the geometry of sight-lines. As we shall soon see, there are many similarities between the models for projective and conformal geometry, many of which can be found in all parabolic model geometries.

However, there are some key differences as well. Perhaps foremost among these differences is that the model for conformal geometry is significantly less obvious than in projective geometry. Recall that a *conformal structure* on a manifold M is an equivalence class $[\mathrm{g}]$, corresponding to all Riemannian metrics of the form $f\mathrm{g}$ for some (smooth) function $f : M \to \mathbb{R}_+$. Deciding what should count as the stabilizer of this structure is a bit trickier than just preserving sight-lines, especially if we don't necessarily know what the base manifold should be either.

Fortunately, this difficulty also gives us a convenient opportunity to showcase a useful algebraic construction called *Tanaka prolongation*, which happens to solve this issue. As such, our plan for the lecture is as follows:

- Motivate what Tanaka prolongation does geometrically
- Explain the construction in the case of conformal geometry
- Describe conformal motion from an observer perspective
- (Appendix) Sketch how Tanaka prolongation works in general

By the end of this lecture, we should have a good idea of what the model for conformal geometry looks like. While there are additional nuances to more general parabolic models, many of the main ideas are similar, so the reader will hopefully be prepared to encounter other parabolic geometries on their own. In the next lecture, we will finally see what a Cartan connection is, and why they're so easy to work with.

1. From similarity to conformality

Let us imagine that we don't already know what our model (G, P) for conformal geometry should be. Where's a good place to start exploring what this model could be?

We're looking for a Lie group G corresponding to symmetries that preserve a Riemannian metric up to scale. As such, a good initial candidate would be the Lie group of similarity transformations of $\mathbb{R}^m$.

A *similarity transformation* of $\mathbb{R}^m$ is an affine transformation that preserves the underlying Euclidean metric up to scale. The Lie group of such transformations is isomorphic to $\mathbb{R}^m \rtimes \mathbb{R}_+ \mathrm{O}(m)$, where $\mathbb{R}_+ \mathrm{O}(m)$ is the group of linear transformations that are positive scalar multiples of orthogonal transformations; for $(u, A) \in \mathbb{R}^m \rtimes \mathbb{R}_+ \mathrm{O}(m)$, as with Euclidean geometry, the action on $\mathbb{R}^m$ is just $(u, A) \cdot v = u + A(v)$. The geometry of the model $(\mathbb{R}^m \rtimes \mathbb{R}_+ \mathrm{O}(m), \mathbb{R}_+ \mathrm{O}(m))$ is called *similarity geometry*.

This geometry looks a lot like Euclidean geometry from an observer perspective. Thinking of $\mathbb{R}^m \rtimes \mathbb{R}_+ \mathrm{O}(m) \simeq \mathrm{I}(m) \rtimes \mathbb{R}_+$ as a bundle of perspectives for ourselves as observers within the geometry, it's essentially the same as the Euclidean case except that we can now also right-translate by elements of the subgroup $\mathbb{R}_+\mathbb{1}$ to rescale ourselves.

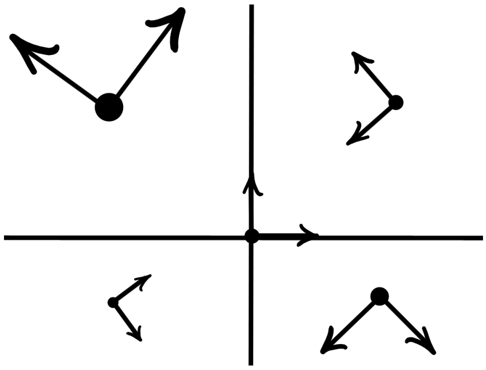

FIGURE 1. As with Euclidean geometry, we can think of $\mathbb{R}^m \rtimes \mathbb{R}_+ \mathrm{O}(m)$ as a bundle of perspectives for ourselves as observers inside similarity geometry

Let us suggestively denote by G_- the subgroup of translations $\mathbb{R}^m$, and by G_0 the subgroup $\mathbb{R}_+ \mathrm{O}(m)$, so that the model for similarity geometry is (G_-G_0, G_0).

Inside the Lie subalgebra $\mathfrak{g}_0$, we can preemptively denote by E_{gr} the element with one-parameter subgroup $\exp(tE_{\mathrm{gr}}) = e^{-t}\mathbb{1}$; note that $\mathrm{ad}_{E_{\mathrm{gr}}}$ restricts to multiplication by -1 on $\mathfrak{g}_-$ and vanishes on $\mathfrak{g}_0$. We can also define a homomorphism $\lambda : G_0 \to \mathbb{R}_+$ given by $rA \mapsto |r|$ for $A \in \mathrm{O}(m)$ and $r \in \mathbb{R}^\times$. The kernel of this homomorphism is just $\mathrm{O}(m)$, and we can decompose G_0 as $\exp(\mathbb{R}E_{\mathrm{gr}}) \ker(\lambda) = \mathbb{R}_+ \mathrm{O}(m)$.

Fixing an inner product g_0 on $T_0\mathbb{R}^m$, corresponding to the "usual" one for Euclidean geometry, we can determine a new inner product $\varphi \cdot \mathrm{g}_0$ on $T_{\varphi(0)}\mathbb{R}^m = T_{q_{G_0}(\varphi)}(G_-G_0/G_0)$ for each $\varphi \in G_-G_0$ by

$$\varphi \cdot \mathrm{g}_0(v, w) := \mathrm{g}_0(\varphi_*^{-1}(v), \varphi_*^{-1}(w)).$$

In particular, the "usual" Riemannian metric for Euclidean geometry is given by $\mathrm{g}_x := x \cdot \mathrm{g}_0$ for each $x \in \mathbb{R}^m = G_-$.

Since G_- is a normal subgroup of G_-G_0 and $G_- \cap G_0 = \{e\}$, we have a natural quotient homomorphism $\pi_{G_-} : G_-G_0 \to G_0$. When used together with the homomorphism $\lambda : G_0 \to \mathbb{R}_+$, this gives us a convenient way of describing the inner product $\varphi \cdot \mathrm{g}_0$ for arbitrary $\varphi \in G_-G_0$: the subgroup $G_- \ker(\lambda) = \mathrm{I}(m)$ acts by isometries, so

$$\varphi \cdot \mathrm{g}_0 = \lambda(\pi_{G_-}(\varphi))^{-2}\mathrm{g}_{\varphi(0)}.$$

Because a Riemannian metric $\tilde{\mathrm{g}}$ conformal to g is, by definition, of the form $\tilde{\mathrm{g}} = f\mathrm{g}$ for some (smooth) function $f : \mathbb{R}^m \to \mathbb{R}_+$, we can identify a choice of metric conformal to the Euclidean one with a choice of section $\sigma_f : G_-G_0/G_0 \to G_-G_0/\ker(\lambda)$. Explicitly, for each $x \in G_-$, we define $\sigma_f(x) := x(\lambda^{-1}(\sqrt{f(x)}))^{-1}$, so that

$$\sigma_f(x) \cdot \mathrm{g}_0 = \lambda(\lambda^{-1}(\sqrt{f(x)})^{-1})^{-2}\mathrm{g}_x = f(x)\mathrm{g}_x = \tilde{\mathrm{g}}_x.$$

In other words, similar to what we saw with projective geometry last time, we can think of the conformal structure on the base manifold G_-G_0/G_0 as something that "lives" in the principal $\exp(\mathbb{R}E_{\mathrm{gr}})$-bundle $G_-G_0/\ker(\lambda)$ over G_-G_0/G_0.

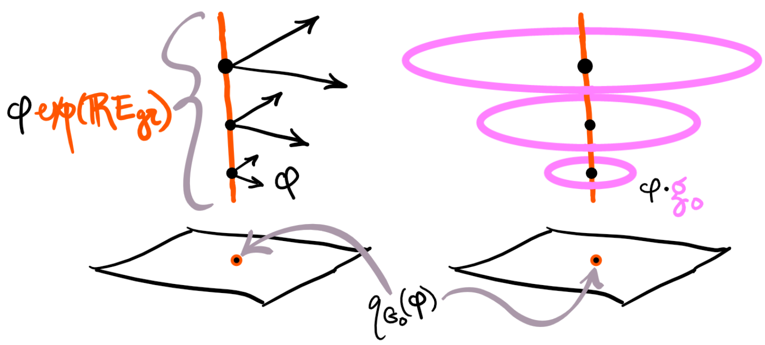

FIGURE 2. Different elements of $G_-G_0/\ker(\lambda)$ over a given point in G_-G_0/G_0 correspond to different choices of inner product over that point, depicted here as the unit disks determined by these inner products

Intuitively, this means that the fibers of $G_-G_0/\ker(\lambda)$ over the base manifold G_-G_0/G_0 are like the sight-lines from projective geometry. Indeed, the key invariant of conformal geometry is the choice of metric up to scale, and by the above, these fibers essentially correspond to the space of all such choices over a given point, so it makes sense that these are what the geometry wants to keep preserved.

Now, we want to give ourselves a notion of "conformal frame", so that we can meaningfully place ourselves inside of this geometry. As with projective geometry, it is convenient to build up these frames in steps. To start, we consider the principal $\exp(\mathbb{R}E_{\mathrm{gr}})$-bundle $G_-G_0/\ker(\lambda)$ over G_-G_0/G_0, which we think of as the space where the conformal structure actually lives and whose elements correspond to choices of scale for the Euclidean metric at the underlying point on the base manifold. From here, we can naturally include the orthonormal frames for the metric at each choice of scale; this amounts to moving up to the principal $\ker(\lambda)$-bundle G_-G_0 over the space $G_-G_0/\ker(\lambda)$, which (unsurprisingly) makes G_-G_0 into a principal G_0-bundle over the base manifold G_-G_0/G_0. Here, $\ker(\lambda)$ accounts for stabilizer motion that preserves the metric and scale, and $\exp(\mathbb{R}E_{\mathrm{gr}})$ lets us rescale directly; neither of these changes the fact that motion from G_- preserves the scale. Thus, the final step is to include "higher-order frames" corresponding to changes of perspective where motion from G_- can alter the scale.

This leads to an obvious question: what can such "higher-order frames" be?

2. Tanaka prolongation in the conformal case

Recall that, given a parabolic model (G, P), we can often think of the geometry of (G_-G_0, G_0) as a kind of affine analogue of the geometry of (G, P). *Tanaka prolongation* gives a way to reverse this analogy, so that given the "affine version", we can (usually) build the corresponding parabolic structure.

Back in the conformal case, let's work at the level of Lie algebras. We already have a Lie algebra $\mathfrak{g}_- + \mathfrak{g}_0$, where $\mathfrak{g}_-$ is the subalgebra of translations $\mathbb{R}^m$ and $\mathfrak{g}_0$ is the subalgebra $\mathbb{R}\mathbb{1} + \mathfrak{o}(m)$. Writing $\mathfrak{g}_{-1} := \mathfrak{g}_-$ and $\mathfrak{g}_{-\ell} := \{0\}$ for all $\ell > 1$, this gives a (somewhat boring) graded Lie algebra structure on $\mathfrak{g}_- + \mathfrak{g}_0$: for all $i, j \leq 0$, we have $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. Algebraically, the goal of Tanaka prolongation is to extend this to a new graded Lie algebra $\mathfrak{g} := \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots$. From the above, we know that the subalgebra $\sum_{\ell>0} \mathfrak{g}_\ell$, which we will preemptively call $\mathfrak{p}_+$, should correspond to changes of perspective that allow motion from $\mathfrak{g}_-$ to change the scale.

To build this new Lie algebra, let us start by considering what $\mathfrak{g}_1$ must do to $\mathfrak{g}_-$: to maintain the graded structure, we need to have $[\mathfrak{g}_1, \mathfrak{g}_{-1}] \subseteq \mathfrak{g}_0$. Moreover, because we want the end result to be a Lie algebra, it should satisfy the Jacobi identity, so for $v, w \in \mathfrak{g}_{-1}$ and $\alpha \in \mathfrak{g}_1$, we should have

$$0 = [\alpha, [v, w]] = [[\alpha, v], w] + [v, [\alpha, w]].$$

Deciding[1] that $\alpha \in \mathfrak{g}_1$ should be uniquely determined by the action of ad_α on $\mathfrak{g}_-$, we can therefore identify all of the possible choices for elements of $\mathfrak{g}_1$ with the space of linear maps $\alpha \in \mathfrak{g}_-^\vee \otimes \mathfrak{g}_0$ such that $[\alpha(v), w] + [v, \alpha(w)] = \alpha(v)w - \alpha(w)v = 0$ for all $v, w \in \mathfrak{g}_-$.

[1]Adding in central elements isn't interesting in this case; if $[\alpha, \mathfrak{g}_{-1}] = \{0\}$, then it isn't doing anything.

For context, let us give some algebraic definitions. Given an arbitrary Lie algebra $\mathfrak{h}$ and $\mathfrak{h}$-representation V, define

$$\partial : \mathfrak{h}^\vee \otimes V \to \Lambda^2\mathfrak{h}^\vee \otimes V$$

by $\partial\alpha(X \wedge Y) := X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha([X, Y])$. With this, we can further define the space $\mathrm{Der}(\mathfrak{h}; V) := \{\alpha \in \mathfrak{h}^\vee \otimes V : \partial(\alpha) = 0\}$ of *derivations* from $\mathfrak{h}$ to V. In our case, we are identifying $\mathfrak{g}_1$ with the subspace of $\mathrm{Der}(\mathfrak{g}_-; \mathfrak{g}_- + \mathfrak{g}_0)$ with images contained in $\mathfrak{g}_0$.

The space of all linear maps $\alpha : \mathfrak{g}_- \to \mathfrak{g}_0$ satisfying

$$\alpha(v)w - \alpha(w)v = 0$$

for all $v, w \in \mathfrak{g}_-$ is determined by the $\dim(\mathfrak{g}_-^\vee)\dim(\Lambda^2\mathfrak{g}_-)$ independent linear equations

$$\mathrm{g}_0(e_k, \alpha(e_i)e_j - \alpha(e_j)e_i) = 0.$$

In other words, we can identify the component $\mathfrak{g}_1$ with a subspace of $\mathfrak{g}_-^\vee \otimes \mathfrak{g}_0$ of dimension

$$\begin{aligned}\dim(\mathfrak{g}_1) &= \dim(\mathfrak{g}_-^\vee \otimes \mathfrak{g}_0) - \dim(\mathfrak{g}_-^\vee)\dim(\Lambda^2\mathfrak{g}_-)\\ &= \dim(\mathfrak{g}_-^\vee)(\dim(\mathfrak{g}_0) - \dim(\Lambda^2\mathfrak{g}_-))\\ &= \dim(\mathfrak{g}_-^\vee).\end{aligned}$$

Indeed, it turns out there is a convenient identification between $\mathfrak{g}_-^\vee$ and $\mathfrak{g}_1$: for $\alpha \in \mathfrak{g}_-^\vee$, the corresponding map $\mathrm{ad}_\alpha|_{\mathfrak{g}_-}$ is given by

$$v \mapsto -\alpha(v)\mathbb{1} - \alpha \otimes v + \mathrm{g}_0(v, \cdot) \otimes \alpha^\sharp,$$

where $\alpha^\sharp \in \mathfrak{g}_-$ is the unique element such that $\mathrm{g}_0(\alpha^\sharp, \cdot) = \alpha$. In other words, $\mathrm{ad}_\alpha(v)w = -\alpha(v)w - \alpha(w)v + \mathrm{g}_0(v, w)\alpha^\sharp$.

Now that we have $\mathfrak{g}_1$, we can try the same thing with $\mathfrak{g}_2$. We want elements $\beta \in \mathfrak{g}_2$ to satisfy $0 = \beta([v, w]) = [\beta(v), w] + [v, \beta(w)]$, with $\beta(v), \beta(w) \in \mathfrak{g}_1 \approx \mathfrak{g}_-^\vee$ to preserve the graded structure. From here, the computations get a bit heinous, but the key thing to note is that, when $\dim(\mathfrak{g}_-) > 2$, we must have $\mathfrak{g}_2 = \{0\}$, so the construction stops with $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. Thinking of $\alpha \in \mathfrak{g}_1$ as a linear map from $\mathfrak{g}_-$ to $\mathfrak{g}_0$, we define

$$[\alpha, R] = \alpha \circ R - \mathrm{ad}_R \circ \alpha$$

for each $R \in \mathfrak{g}_0$ and define $\mathfrak{g}_1$ to be abelian. This gives $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ a graded Lie algebra structure, and a faithful representation of $\mathfrak{g}$ is given by

$$(v, r\mathbb{1} + R, \alpha) \mapsto \begin{bmatrix} -r & (\alpha^\sharp)^\top & 0 \\ v & R & -\alpha^\sharp \\ 0 & -v^\top & r \end{bmatrix}.$$

Letting G be the Lie group with Lie algebra $\mathfrak{g}$ such that $G_-G_0 \leq G$ and G/G_0 is connected, we can define $P := G_0P_+$ for P_+ the connected subgroup generated by $\mathfrak{p}_+ = \mathfrak{g}_1$, and our model for conformal geometry becomes (G, P). It is not too difficult to check that G is isomorphic to $\mathrm{PO}(1, m+1)$, with corresponding quadratic form Q on $\mathbb{R}^{m+2}$ given by

$$Q\left(\begin{bmatrix} x_0 \\ x \\ x_{m+1} \end{bmatrix}\right) = -2x_0x_{m+1} + \sum_{i=1}^{m} x_i^2.$$

What about when $\dim(\mathfrak{g}_-) = m = 2$? Well, in that case, $\mathfrak{g}_2$, and more generally each $\mathfrak{g}_\ell$ with $\ell > 1$, is not trivial, so the $\mathfrak{g}$ we construct is infinite-dimensional. This is one of the issues with Tanaka prolongation: sometimes, the information you put into it isn't sufficient to return a finite-dimensional Lie algebra. We could probably have expected some sort of problem here, though; in dimension two, all holomorphic maps are conformal wherever their derivatives don't vanish, so we were never going to construct a finite-dimensional model symmetry algebra for two-dimensional conformal geometry.

3. Moving around in the conformal sphere

Thinking of G as the Lie group of $\mathrm{PO}(\mathbb{R}^{m+2}, Q)$, with Q as in the previous section, we can see that G acts transitively on the projectivized null-cone $\{\langle x\rangle \in \mathbb{RP}^{m+1} : Q(x) = 0\}$, and that

$$\mathrm{Stab}_G\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left\{\begin{pmatrix} a & p & -p(p^\top)/2 \\ 0 & A & -p^\top \\ 0 & 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^\times,\ p^\top \in \mathbb{R}^m,\ A \in \mathrm{O}(m)\right\}$$

corresponds to the closed subgroup P, so we can identify G/P with the projectivized null-cone.

Topologically, this projectivized null-cone is a sphere. To see this, let $x \in Q^{-1}(0) \setminus \{0\}$. We're trying to understand the projectivized null-cone, so we only care about x up to scale. In particular, we can rescale x so that $x_0 + x_{m+1} = \sqrt{2}$. Under this rescaling, we can define $y := \frac{x_0 - x_{m+1}}{\sqrt{2}}$, so that

$$1 + y = \frac{(x_0 + x_{m+1}) + (x_0 - x_{m+1})}{\sqrt{2}} = \sqrt{2}x_0$$

and

$$1 - y = \frac{(x_0 + x_{m+1}) - (x_0 - x_{m+1})}{\sqrt{2}} = \sqrt{2}x_{m+1}.$$

Thus, $1 - y^2 = (1+y)(1-y) = 2x_0x_{m+1}$, so

$$Q(x) = -2x_0x_{m+1} + \sum_{i=1}^{m} x_i^2 = -1 + y^2 + \sum_{i=1}^{m} x_i^2 = 0,$$

hence we can identify the projectivized null-cone with the space with $y^2 + \sum_{i=1}^{m} x_i^2 = 1$, namely the m-sphere. Because of this, we call G/P the *conformal sphere*.

Alternatively, we can think of G/P as the one-point compactification of $G_-G_0/G_0 \cong \mathbb{R}^m$. The subgroup G_- takes the form

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ v & \mathbb{1} & 0 \\ -v^\top v/2 & -v^\top & 1 \end{pmatrix} : v \in \mathbb{R}^m \right\},$$

so

$$G_- \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left\{ \begin{pmatrix} 1 \\ v \\ -v^\top v/2 \end{pmatrix} : v \in \mathbb{R}^m \right\}$$

gives us our open cell. The complement of this open cell inside the projectivized null-cone is the single point $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$, which we can think of as the "point at infinity" for the copy of Euclidean space given by the open cell.

As for movement within G, things are much the same as with projective geometry. At each configuration $g \in G$, we determine a copy $q_P(gG_-)$ of Euclidean space corresponding to motion from G_-. Using $\exp(\mathbb{R}E_{\mathrm{gr}})$, we can control the scale of that copy of Euclidean space, and with $\ker(\lambda)$, we can move amongst the different orthonormal frames for that point in Euclidean space. Finally, the subgroup P_+ gives us "unipotent tilts", which let us tilt between different copies of Euclidean space through our underlying point $q_P(g) \in G/P$.

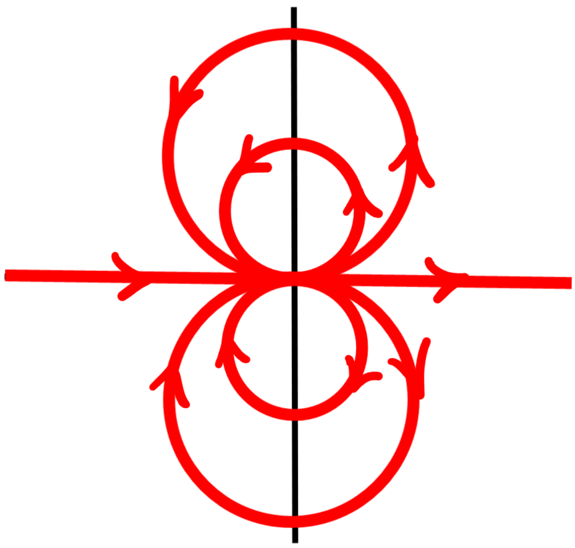

FIGURE 3. In conformal geometry, the trajectories of motion from one-parameter subgroups in G_- are not uniquely determined by an initial velocity in the base manifold

Perhaps the main difference here is that motion from G_- no longer determines consistent curves on the base manifold up to reparametrization. However, again, the motion is consistent and meaningful inside of G.

Appendix: Tanaka prolongation in general

Both conformal and projective geometry are $|1|$-graded, meaning that the grading of $\mathfrak{g}$ determined by a Cartan involution θ and the parabolic $\mathfrak{p}$ is of the form $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. This makes many algebraic aspects of these geometries fairly simplistic compared to the general case. In particular, Tanaka prolongation is a bit more involved when there are multiple negative grading components.

Let's imagine we have a graded nilpotent Lie algebra

$$\mathfrak{g}_- = \mathfrak{g}_{-k} + \cdots + \mathfrak{g}_{-1},$$

with $[\mathfrak{g}_{-i}, \mathfrak{g}_{-j}] \subseteq \mathfrak{g}_{-i-j}$, together with another Lie algebra $\mathfrak{g}_0$ that both acts on $\mathfrak{g}_-$ by derivations—so $R \cdot [v, w] = [R \cdot v, w] + [v, R \cdot w]$ for all $R \in \mathfrak{g}_0$ and $v, w \in \mathfrak{g}_-$—and preserves the grading—so $R \cdot \mathfrak{g}_{-i} \subseteq \mathfrak{g}_{-i}$ for every $R \in \mathfrak{g}_0$ and each $i \geq 1$. We consider the semidirect sum $\mathfrak{g}_- \niplus \mathfrak{g}_0$, which we will write as just $\mathfrak{g}_- + \mathfrak{g}_0$; what is the corresponding Tanaka prolongation?

In essence, we follow the same idea as before: build up positive grading components piece by piece. To start, we want $\mathfrak{g}_1$, consisting of elements that act as derivations that send each $\mathfrak{g}_{-i}$ to $\mathfrak{g}_{-i+1}$. In other words, we are looking for

$$\mathfrak{g}_1 := \{\alpha \in \operatorname{Der}(\mathfrak{g}_-; \mathfrak{g}_- + \mathfrak{g}_0) : \alpha(\mathfrak{g}_{-i}) \subseteq \mathfrak{g}_{-i+1} \text{ for each } i > 0\}.$$

Naturally, $\mathfrak{g}_- + \mathfrak{g}_0 + \mathfrak{g}_1$ is a representation of $\mathfrak{g}_- + \mathfrak{g}_0$: given $\alpha \in \mathfrak{g}_1$, we have $v \cdot \alpha = -\alpha(v)$ for $v \in \mathfrak{g}_-$ and $R \cdot \alpha = \operatorname{ad}_R \circ \alpha - \alpha \circ \operatorname{ad}_R$ for $R \in \mathfrak{g}_0$.

Next, we want to build a grading component $\mathfrak{g}_2$ of degree 2, so that its elements act as derivations sending each $\mathfrak{g}_{-i}$ to $\mathfrak{g}_{-i+2}$. Symbolically, this means

$$\mathfrak{g}_2 := \{\beta \in \operatorname{Der}(\mathfrak{g}_-; \mathfrak{g}_- + \mathfrak{g}_0 + \mathfrak{g}_1) : \beta(\mathfrak{g}_{-i}) \subseteq \mathfrak{g}_{-i+2} \text{ for each } i > 0\}.$$

Again, $\mathfrak{g}_- + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$ is a representation of $\mathfrak{g}_- + \mathfrak{g}_0$: for $\beta \in \mathfrak{g}_2$, $v \cdot \beta = -\beta(v)$ and $R \cdot \beta = \operatorname{ad}_R \circ \beta - \beta \circ \operatorname{ad}_R$ as before.

We continue this process, recursively defining

$$\mathfrak{g}_\ell := \left\{\zeta \in \operatorname{Der}\left(\mathfrak{g}_-; \sum_{j<\ell} \mathfrak{g}_j\right) : \zeta(\mathfrak{g}_{-i}) \subseteq \mathfrak{g}_{-i+\ell} \text{ for each } i > 0\right\}$$

for each $\ell > 0$. Letting $\mathfrak{g}$ be the representation of $\mathfrak{g}_- + \mathfrak{g}_0$ given by the sum

$$\mathfrak{g} := \mathfrak{g}_- + \mathfrak{g}_0 + \sum_{\ell>0} \mathfrak{g}_\ell$$

of all of these grading components, we imbue it with a Lie algebra structure as follows. First, the bracket agrees with the bracket of $\mathfrak{g}_- + \mathfrak{g}_0$ when restricted there, and for each $\alpha \in \sum_{\ell>0} \mathfrak{g}_\ell$ and $X \in \mathfrak{g}_- + \mathfrak{g}_0$, $[X, \alpha] := X \cdot \alpha$, where $\cdot$ denotes the representation action of $\mathfrak{g}_- + \mathfrak{g}_0$ on $\mathfrak{g}$. From here, we want to continue defining the bracket in a way that satisfies the Jacobi identity, so that

$$[\alpha, \beta](v) = [[\alpha, \beta], v] = [[\alpha, v], \beta] + [\alpha, [\beta, v]] = [\alpha(v), \beta] + [\alpha, \beta(v)]$$

for $\alpha, \beta \in \sum_{\ell>0} \mathfrak{g}_\ell$ and $v \in \mathfrak{g}_-$. Conveniently, this gives us a way to construct the bracket recursively as well: for $\alpha, \alpha' \in \mathfrak{g}_1$, we define $[\alpha, \alpha'] \in \mathfrak{g}_2 \subseteq \mathrm{Der}(\mathfrak{g}_-; \mathfrak{g}_- + \mathfrak{g}_0 + \mathfrak{g}_1)$ to be the unique element of the form

$$[\alpha, \alpha'](v) = [\alpha(v), \alpha'] + [\alpha, \alpha'(v)]$$

for each $v \in \mathfrak{g}_-$. Since $\alpha(v), \alpha'(v) \in \mathfrak{g}_- + \mathfrak{g}_0$, we already know that $[\alpha(v), \alpha'] = \alpha(v) \cdot \alpha'$ and $[\alpha, \alpha'(v)] = -\alpha'(v) \cdot \alpha$, so this bracket is well-defined. Then, for arbitrary $\beta \in \mathfrak{g}_i$ and $\zeta \in \mathfrak{g}_j$, we can recursively define $[\beta, \zeta] \in \mathfrak{g}_{i+j}$ to be the unique element such that

$$[\beta, \zeta](v) = [\beta(v), \zeta] + [\beta, \zeta(v)]$$

for each $v \in \mathfrak{g}_-$. Since $\beta(v) \in \sum_{\ell<i} \mathfrak{g}_\ell$ and $\zeta(v) \in \sum_{\ell<j} \mathfrak{g}_\ell$, if we know how to form brackets with elements in grading components of lesser degree, then these brackets are well-defined as well.

Thus, we get a Lie algebra $\mathfrak{g}$. If $\mathfrak{g}_\ell = \{0\}$ for every ℓ greater than some k, then $\mathfrak{g}$ is finite-dimensional and $\mathfrak{p}_+ := \sum_{\ell>0} \mathfrak{g}_\ell$ is a nilpotent subalgebra. Because $[\mathfrak{g}_i, [\mathfrak{g}_j, \mathfrak{g}_\ell]] \subseteq \mathfrak{g}_{i+j+\ell}$, we must have $\mathfrak{Z}(\mathfrak{g}_i, \mathfrak{g}_j) = \{0\}$ unless $i + j = 0$, so if $\mathfrak{g}$ is semisimple, then $\mathfrak{g}_-$ and $\mathfrak{p}_+$ must be $\mathfrak{Z}$-dual. In particular, if $\mathfrak{g}$ is semisimple, then $\mathfrak{p} := \mathfrak{g}_0 + \mathfrak{p}_+$ must satisfy $\mathfrak{p}^\perp = \mathfrak{p}_+$, hence $\mathfrak{p}$ must be parabolic.